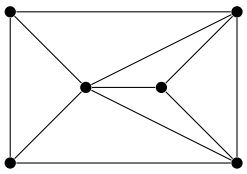
Instructions: Legibly complete each of the following on lined paper and submit on Gradescope.

1. Prove the complete graph K_n is connected for all $n \geq 1$.
2. Prove the complete bipartite graph $K_{m,n}$ is connected for all $m, n \geq 1$.
3. Let G be the graph depicted below.



   (a) Does G have a Hamilton cycle? Prove it.

   (b) Does G have a Hamilton path? Prove it.

   (c) Does G have an Euler trail? Prove it.

   (d) Compute the chromatic number of G (with proof).

4. Let G be a simple graph and $v \notin V(G)$. Obtain a new graph $v * G$ from G with vertices $V(v * G) = V(G) \cup \{v\}$ and edges $E(v * G) = E(G) \cup \{\{v, w\} : w \in V(G)\}$. In particular, $v * G$ is the graph obtained by adding a new vertex to G and adding all possible edges from the new vertex to the old vertices.

   (a) Draw $v * C_n$ for $n \in \{3, 4, 7\}$.

   (b) Prove $K_{n+1} = (n + 1) * K_n$ for all $n \in \mathbb{Z}_+$.

   (c) Prove $\chi(v * G) = \chi(G) + 1$ for all graphs G.